Q4 QAM with Phase and Frequency Offset

Sunday, July 03, 2011
7:26 PM

$$x_{QAM}(t) = m_1(t)\sqrt{2}\cos(\omega_c t) + m_2(t)\sin(\omega_c t)$$

Recall the product-to-sum formula:

$$\cos A \cos B = \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right).$$

In this question, we will need two more formulas involving products of sine and cosine functions.

Again, we will use the Euler's identity:

$$\sin A = \frac{1}{2j}\left(e^{jA} - e^{-jA}\right)$$

$$\cos B = \frac{1}{2}\left(e^{jB} + e^{-jB}\right)$$

Hence,

$$\sin A \cos B = \frac{1}{4j}\left(e^{jA} - e^{-jA}\right)\left(e^{jB} + e^{-jB}\right)$$

$$= \frac{1}{4j}\left(e^{j(A+B)} - e^{-j(A+B)} + e^{j(A-B)} - e^{-j(A-B)}\right)$$

$$= \frac{1}{4j}\left(2j\sin(A+B) + 2j\sin(A-B)\right)$$

$$= \frac{1}{2}\sin(A+B) + \frac{1}{2}\sin(A-B),$$

and

$$\sin A \sin B = \frac{1}{(2j)^2}\left(e^{jA} - e^{-jA}\right)\left(e^{jB} - e^{-jB}\right)$$

$$= \frac{1}{-4}\left(e^{j(A+B)} + e^{-j(A+B)} - e^{j(A-B)} - e^{-j(A-B)}\right)$$

$$= -\frac{1}{4}\left(2\cos(A+B) - 2\cos(A-B)\right)$$

$$= \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

(a) Let $v_1(t) = x_{QAM}(t)\sqrt{2}\cos\left((\omega_c+\Delta\omega)t+\delta\right)$

and $\hat{m}_1(t) = \text{LPF}\{v_1(t)\}$.

Then, by the product-to-sum formulas,

$$v_1(t) = m_1(t)\cos\left(\underbrace{(2\omega_c+\Delta\omega)t+\delta}_{\text{LPF} \to 0}\right) + m_1(t)\cos\left((\Delta\omega)t+\delta\right)$$

$$+ m_2(t)\sin\left(\underbrace{(2\omega_c+\Delta\omega)t+\delta}_{\text{LPF} \to 0}\right) + m_2(t)\sin\left(-(\Delta\omega)t-\delta\right)$$

$$\hat{m}_1(t) = m_1(t)\cos\left((\Delta\omega)t+\delta\right) - m_2(t)\sin\left((\Delta\omega)t+\delta\right)$$

(b) Let $v_2(t) = x_{QAM}(t)\sqrt{2}\sin\left((\omega_c+\Delta\omega)t+\delta\right)$

and $\hat{m}_2(t) = \text{LPF}\{v_2(t)\}$.

Then, by the product-to-sum formulas,

$$v_2(t) = m_1(t)\sin\left(\underbrace{(2\omega_c+\Delta\omega)t+\delta}_{\text{LPF} \to 0}\right) + m_1(t)\sin\left((\Delta\omega)t+\delta\right)$$

$$+ m_2(t)\cos\left((\Delta\omega)t+\delta\right) - m_2(t)\cos\left(\underbrace{(2\omega_c+\Delta\omega)t+\delta}_{\text{LPF} \to 0}\right)$$

$$\hat{m}_2(t) = m_1(t)\sin\left((\Delta\omega)t+\delta\right) + m_2(t)\cos\left((\Delta\omega)t+\delta\right)$$

Q5 (a) Time Delay and Phase Offset (b) HWR Rx with Time Delay

Thursday, November 11, 2010
11:17 AM

(a)

Since the modification is made at the receiver, any results before it is unchanged and therefore we still have

$$x(t-\tau) \longrightarrow \otimes \xrightarrow{v(t)} \boxed{\text{LPF}} \longrightarrow \hat{m}(t)$$

$$\uparrow \sqrt{2}\cos(\omega_c t - \theta)$$

where

$$x(t-\tau) = m(t-\tau)\sqrt{2}\cos(\omega_c t - \underbrace{\omega_c \tau}_{\equiv \phi \text{ as defined in lecture.}})$$

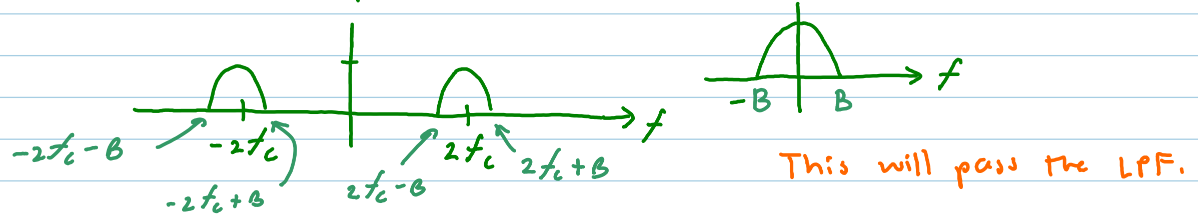
Let $v(t)$ be the signal before the LPF.

Then $v(t) = x(t-\tau) \times \sqrt{2}\cos(\omega_c t - \theta)$

$$= 2\, m(t-\tau)\cos(\omega_c t - \phi)\cos(\omega_c t - \theta)$$

$$= m(t-\tau)\big(\cos(2\omega_c t - \phi - \theta) + \cos(\theta - \phi)\big)$$

$$= \underbrace{m(t-\tau)\cos(2\omega_c t - \phi - \theta)}_{\mathcal{F}} + \underbrace{m(t-\tau)}_{\mathcal{F}}\underbrace{\cos(\theta - \phi)}_{\text{a constant}}$$



$$\hat{m}(t) = m(t-\tau)\cos(\theta - \phi) = m(t-\tau)\cos(\theta - \omega_c\tau).$$

(b)

Again, we have

$$x(t-\tau) = m(t-\tau)\sqrt{2}\cos(\omega_c(t-\tau))$$

$$x(t-\tau) \longrightarrow \boxed{\text{HWR}} \xrightarrow{v(t)} \boxed{\text{LPF}} \longrightarrow \hat{m}(t)$$

Let $v(t)$ be the signal before the LPF.

Then, $v(t) = x(t-\tau) \times \underbrace{r(t-\tau)}$, where $r(t) = 1[\cos(2\pi f_c t) \geq 0]$

$\hookrightarrow$ Because $m(t-\tau)$ is always ≥ 0, the sign of $x(t-\tau)$ only depends on $\cos(\omega_c(t-\tau))$, which is simply a shifted version of $\cos(\omega_c t)$.

All of the analysis is the same as what was presented in class except that we now have a time shift of amount τ.

Recall that

$$r(t) = \frac{1}{2} + \frac{2}{\pi}\cos\omega_c t - \frac{2}{\pi}\times\frac{1}{3}\cos 3\omega_c t + \ldots$$

$$= \sum_{k=0}^{\infty} a_k \cos(k\omega_c t)$$

where $a_0 = \frac{1}{2}$, $a_1 = \frac{2}{\pi}$, $a_2 = 0$, $a_3 = \frac{2}{\pi}\times\frac{1}{3}$, ...

We then have

$$v(t) = x(t-\tau) \times r(t-\tau)$$

$$= m(t-\tau)\sqrt{2}\cos(\omega_c(t-\tau)) \sum_{k=0}^{\infty} a_k \cos(k\omega_c(t-\tau))$$

$$= \sqrt{2}\, m(t-\tau) \sum_{k=0}^{\infty} a_k \cos(\omega_c(t-\tau))\cos(k\omega_c(t-\tau))$$

$$= \sqrt{2}\, m(t-\tau) \sum_{k=0}^{\infty} \frac{1}{2}a_k\left(\cos((k-1)\omega_c(t-\tau)) + \cos((k+1)\omega_c(t-\tau))\right)$$

So, $v(t)$ will be a linear combination of signals of the form

$$\sqrt{2}\times\frac{1}{2}\times a_k \times m(t-\tau)\cos(n\omega_c(t-\tau))$$

n: $k-1$ or $k+1$

We know that the spectrum of $m(t)\cos(n\omega_c t)$ is the spectrum of $m(t)$ shifted to $\pm 2\pi f_c \times n$ and scaled by $\frac{1}{2}$.

The time shift results in an extra factor of $e^{-j2\pi f\tau}$ which does not affect the location of the spectrum.

Recall that $\hat{m}(t) = LPF\{v(t)\}$.

The only part of $v(t)$ that will pass through the LPF would be the one that is centered around 0 Hz. (DC)

This corresponds to the case when $n=0$

$k-1$ or $k+1$

The corresponding k is $k=1$ or -1.

not in the summation.

Therefore, $\hat{m}(t) = \sqrt{2} \times \frac{1}{2} \times a_1 \times m(t-\tau)$.

For HWR, $a_1 = \frac{2}{\pi}$.

Hence,

$$\hat{m}(t) = \frac{\sqrt{2}}{\pi} m(t-\tau)$$

Q6 FWR Rx with Time Delay

Sunday, August 05, 2012
9:46 PM

(a) Let's start with FWR input-output relation:

$$f_{FWR}(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0. \end{cases}$$

Here, the input is $x(t-\tau)$. So, the output is

$$v(t) = \begin{cases} x(t-\tau), & x(t-\tau) \geq 0 \\ -x(t-\tau), & x(t-\tau) < 0. \end{cases}$$

Now, we know more about the characteristics of $x(t-\tau)$.

In particular, we know that $x(t-\tau) = m(t-\tau)\cos(\omega_c(t-\tau))$

and that $m(t) \geq 0$ at all t

(therefore $m(t-\tau) \geq 0$ at all t.)

The nonnegativity of $m(t)$ means that the sign of $x(t-\tau)$ will depend only on $\cos(\omega_c(t-\tau))$.

Therefore, $$v(t) = \begin{cases} x(t-\tau), & \cos(\omega_c(t-\tau)) \geq 0 \\ -x(t-\tau), & \cos(\omega_c(t-\tau)) < 0. \end{cases}$$

$$= x(t-\tau) \times g_{FWR}(t-\tau)$$

where $$g_{FWR}(t-\tau) = \begin{cases} 1, & \cos(\omega_c(t-\tau)) \geq 0 \\ -1, & \cos(\omega_c(t-\tau)) < 0. \end{cases}$$

In other words, $$g_{FWR}(t) = \begin{cases} 1, & \cos(\omega_c t) \geq 0 \\ -1, & \cos(\omega_c t) < 0. \end{cases}$$

We have seen in the previous HW question that

for HWR, [illegible]

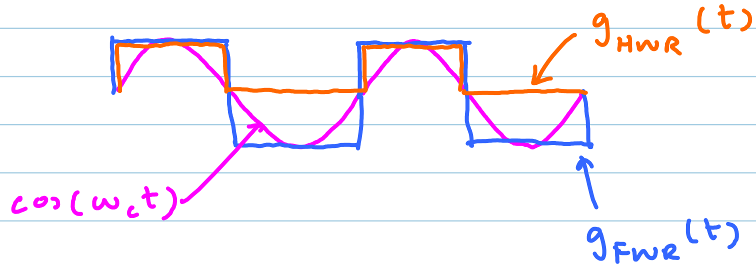
$$v(t) = x(t-\tau) \times 1[\cos(\omega_c(t-\tau)) \geq 0].$$

So, $v(t) = x(t-\tau) \times g_{HWR}(t-\tau)$

where $g_{HWR}(t) = 1[\cos(\omega_c t) \geq 0]$.

The ON-OFF function.

(i) It is easier to find c_1 and c_2 via the plots of g_{FWR} and g_{HWR}.



From the plots, we have $g_{FWR}(t) = 2g_{HWR}(t) - 1$

Therefore, $c_1 = 2$ and $c_2 = -1$

(ii) $$g_{HWR}(t) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t)$$

Therefore, $g_{FWR}(t) = 2g_{HWR}(t) - 1$

$$= \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t).$$

(b) $y(t) = LPF\{v(t)\}$ where

$v(t) = m(t-\tau)\cos(\omega_c(t-\tau))\, g_{FWR}(t-\tau)$.

Let's first consider $v(t+\tau) = m(t)\cos(\omega_c t)\, g_{FWR}(t)$.

$$v(t+\tau) = m(t)\cos(\omega_c t)\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{2k-1}\cos((2k-1)\omega_c t)$$

$$= \frac{2}{\pi}\sum_{k=1}^{\infty}\Big(m(t)\cos((2k-2)\omega_c t) + m(t)\cos(2k\omega_c t)\Big)$$

$k=1$

In freq. domain, these terms will be replica of M(f) shifted to various frequencies.

The only term that shifts to DC is this one at k = 1.

So, $y(t) = \mathrm{LPF}\{v(t)\} = \frac{2}{\pi} m(t-\tau)$.